Model spaces and the compressed shift operator

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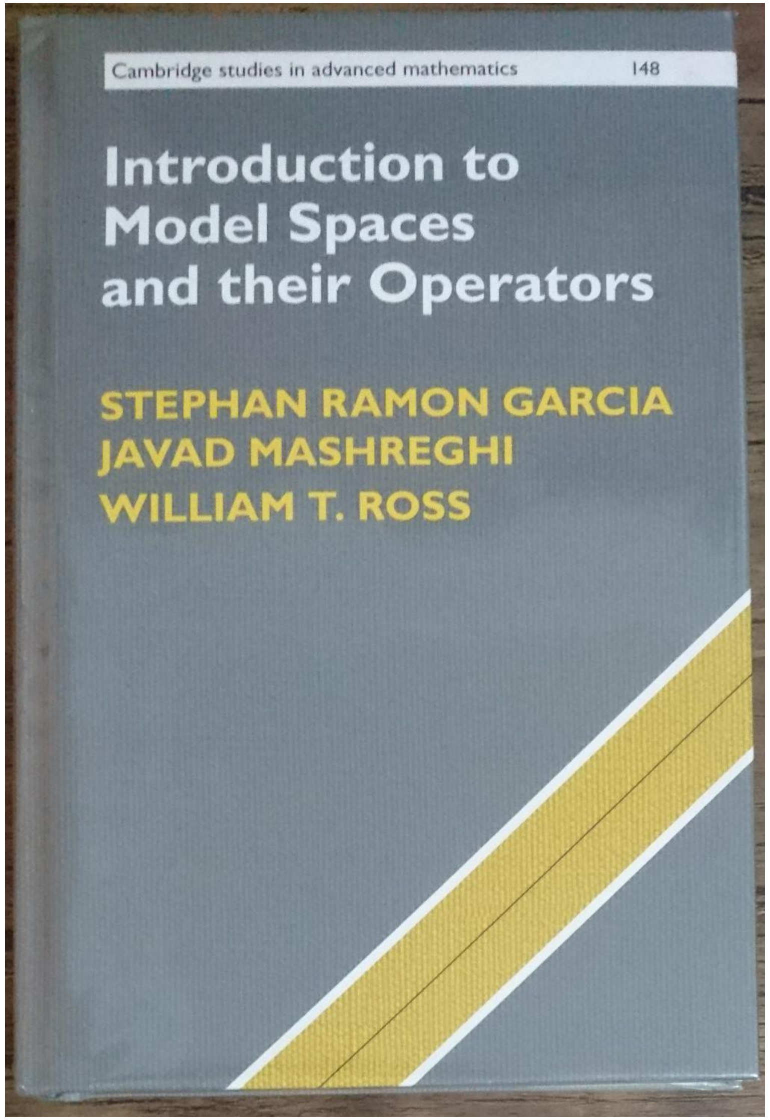
Goals

- Introduce the Hardy space $H^2(\mathbb{D})$ and model space $\mathcal{K}_u = (uH^2)^{\perp}$.

- Introduce key operators on the Hardy space
  - Unilateral (forward) shift $S: H^2 \to H^2$
  - Compressed shift $S_u: \mathcal{K}_u \to \mathcal{K}_u$

- $H^\infty -$ functional calculus $\Lambda: H^\infty \to \mathcal{B}(\mathcal{K}_u)$



Some notations

- Sets
  - $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ $\quad z \in \mathbb{D}$
  - $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ $\quad \zeta \in \mathbb{T}$
- Closed linear span $\bigvee \mathcal{M} = \overline{\text{span}\,\mathcal{M}}$
- Measures
  - $m = \frac{d\lambda}{2\pi}$ normalized Leb meas. on $\mathbb{T}$
  - $M(\mathbb{T}) = \{\text{complex Borel meas.}\}$, $M_+(\mathbb{T}) = \{\text{positive Borel meas.}\}$

The Hardy space $H^2(\mathbb{D})$

The Hardy space $H^2(\mathbb{D})$

- Three equivalent definitions for $H^2 = H^2(\mathbb{D})$

Definition v1 (Taylor coeff.)

$$H^2(\mathbb{D}) = \left\{ f\colon \mathbb{D} \to \mathbb{C} \text{ analytic} : \sum_{n=0}^{\infty} |a_n|^2 < \infty, \text{where } f(z) = \sum_{n=0}^{\infty} a_n z^n \right\}$$

Hilbert space

Definition v2 (Bounded integral means)

$$H^2(\mathbb{D}) = \left\{ f\colon \mathbb{D} \to \mathbb{C} \text{ analytic} : \sup_{0<r<1} \int_{\mathbb{T}} |f(r\zeta)|^2 dm(\zeta) < \infty \right\}$$

Controls growth as we approach $\mathbb{T}$

Definition v3 (Identify with bdry fcts + no negative Fourier coeff.)

$$H^2(\mathbb{D}) = \left\{ f \in L^2(\mathbb{T})\colon \hat{f}(n) = 0 \text{ when } n \le -1, \text{where } f = \sum_{n=-\infty}^{\infty} \hat{f}(n)\zeta^n \right\}$$

Easier to work with bdry fcts / Fourier series

The Hardy space H^2

$$H^2 = \{f\colon \mathbb{D} \to \mathbb{C} \text{ analytic} : \sum |a_n|^2 < \infty, \text{where } f(z) = \sum a_n z^n\}$$
$$= \left\{f\colon \mathbb{D} \to \mathbb{C} \text{ analytic} : \sup_{0<r<1} \int_{\mathbb{T}} |f(r\zeta)|^2 dm(\zeta) < \infty\right\}$$
$$= \left\{f \in L^2(\mathbb{T})\colon \hat{f}(n) = 0 \text{ when } n \le -1, f = \sum \hat{f}(n)\zeta^n\right\}$$

- **Definition v1 (Taylor coeff.)**

$$H^2(\mathbb{D}) = \left\{f\colon \mathbb{D} \to \mathbb{C} \text{ analytic} : \sum_{n=0}^{\infty} |a_n|^2 < \infty, \text{where } f(z) = \sum_{n=0}^{\infty} a_n z^n\right\}$$

Equip this space with inner product

$$\langle f, g\rangle = \left\langle \sum_{n\ge 0} a_n z^n, \sum_{n\ge 0} b_n z^n \right\rangle := \sum_{n\ge 0} a_n \overline{b_n}$$

(write the corresponding norm as $\|\cdot\|$)

So that H^2 becomes a Hilbert space.

The Hardy space H^2

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$$= \left\{f\colon \mathbb{D} \to \mathbb{C} \text{ analytic} : \sup_{0<r<1} \int_{\mathbb{T}} |f(r\zeta)|^2 dm(\zeta) < \infty\right\}$$
$$= \{f \in L^2(\mathbb{T}) \colon \hat{f}(n) = 0 \text{ when } n \leq -1, f = \sum \hat{f}(n)\zeta^n\}$$

- **Definition v2 (Bounded integral means)**

$$H^2(\mathbb{D}) = \left\{f\colon \mathbb{D} \to \mathbb{C} \text{ analytic} : \sup_{0<r<1} \int_{\mathbb{T}} |f(r\zeta)|^2 dm(\zeta) < \infty\right\}$$

Now, using the fact that

$$\int_{\mathbb{T}} |f(r\zeta)|^2 dm(\zeta) = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}$$

We can get

Proposition Let $f\colon \mathbb{D} \to \mathbb{C}$ be analytic. Then $f \in H^2$ iff $\sup_{0<r<1} \int_{\mathbb{T}} |f(r\zeta)|^2 dm(\zeta) < \infty$.

Moreover,

version 1

$$\|f\|^2 = \sup_{0<r<1} \int_{\mathbb{T}} |f(r\zeta)|^2 dm(\zeta)$$

The Hardy space H^2

$$H^2 = \{f\colon \mathbb{D} \to \mathbb{C} \text{ analytic} : \sum |a_n|^2 < \infty, \text{where } f(z) = \sum a_n z^n\}$$
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$$= \left\{f \in L^2(\mathbb{T}) : \hat{f}(n) = 0 \text{ when } n \leq -1, f = \sum \hat{f}(n)\zeta^n\right\}$$

- **Definition v3 (Identify with bdry fcts + no negative Fourier coeff.)**

$$H^2(\mathbb{D}) = \left\{f \in L^2(\mathbb{T}) : \hat{f}(n) = 0 \text{ when } n \leq -1, \text{where } f = \sum_{n=-\infty}^{\infty} \hat{f}(n)\zeta^n\right\}$$

How do we identify $f\colon \mathbb{D} \to \mathbb{C}$ with a function $f\colon \mathbb{T} \to \mathbb{C}$? We need Fatou's theorem

Theorem (Fatou) Let $\mu \in M(\mathbb{T})$ and $\zeta \in \mathbb{T}$. If $(D\mu)(\zeta)$ exists, then

$$\lim_{r\to 1^-} \mathcal{P}(\mu)(r\zeta) = (D\mu)(\zeta)$$

i.e. $\mathcal{P}(\mu)$ has a finite radial limit m-a.e. on $\mathbb{T}$.

Symmetric derivative

Poisson integral $\mathcal{P}(\mu)\colon \mathbb{D} \to \mathbb{C}$

Corollary Let $f = \sum a_n z^n$ in H^2. Then

$$\lim_{r\to 1^-} f(r\zeta) = \lim_{r\to 1^-} \mathcal{P}(f\,dm)(r\zeta) = f(\zeta)$$

exists for m-a.e. $\zeta \in \mathbb{T}$.

The Hardy space H^2

$$H^2 = \{f: \mathbb{D} \to \mathbb{C} \text{ analytic} : \sum |a_n|^2 < \infty, \text{where } f(z) = \sum a_n z^n\}$$
$$= \left\{f: \mathbb{D} \to \mathbb{C} \text{ analytic} : \sup_{0<r<1} \int_{\mathbb{T}} |f(r\zeta)|^2 dm(\zeta) < \infty\right\}$$
$$= \{f \in L^2(\mathbb{T}): \hat{f}(n) = 0 \text{ when } n \leq -1, f = \sum \hat{f}(n)\zeta^n\}$$

- **Definition v3 (Identify with bdry fcts + no negative Fourier coeff.)**

Since $f = \sum a_n z^n \in H^2$, we know a lot more about the bdry fct $\zeta \mapsto f(\zeta)$:

Theorem Let $f = \sum a_n z^n$ in H^2. Then

i. $\lim_{r \to 1^-} f(r\zeta) = f(\zeta)$ exists m-a.e. $\zeta \in \mathbb{T}$.

ii. The boundary fct $\zeta \mapsto f(\zeta)$ belongs to $L^2(\mathbb{T})$

iii. If $n \geq 0$, then $\hat{f}(n) = \mathrm{a_n}$.

If $n \leq -1$, then $\hat{f}(n) = 0$.

Fourier coeffs $\hat{f}(n)$ of the bdry fct $f(\zeta) = \sum a_n \zeta^n$

iv. $\|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2 = \sup_{0<r<1} \int_{\mathbb{T}} |f(r\zeta)|^2 dm(\zeta) = \int_{\mathbb{T}} |f(\zeta)|^2 dm(\zeta)$

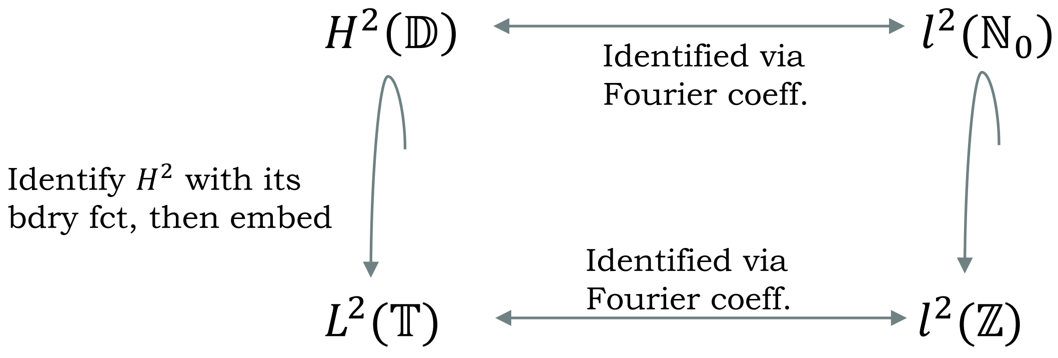
The Hardy space H^2

- **Takeaways**

$$H^2 = \{f : \mathbb{D} \to \mathbb{C} \text{ analytic} : \sum |a_n|^2 < \infty, \text{where } f(z) = \sum a_n z^n\}$$
$$= \left\{f : \mathbb{D} \to \mathbb{C} \text{ analytic} : \sup_{0<r<1} \int_{\mathbb{T}} |f(r\zeta)|^2 dm(\zeta) < \infty\right\}$$
$$= \left\{f \in L^2(\mathbb{T}) : \hat{f}(n) = 0 \text{ when } n \leq -1, f = \sum \hat{f}(n)\zeta^n\right\}$$

via bdry fcts

- We can embed $H^2(\mathbb{D})$ into $L^2(\mathbb{T})$ via its bdry fct, giving us



and $\quad L^2 = H^2 \oplus \overline{\zeta H^2}$

Also, since

$$\|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2 = \sup_{0<r<1} \int_{\mathbb{T}} |f(r\zeta)|^2 dm(\zeta) = \int_{\mathbb{T}} |f(\zeta)|^2 dm(\zeta),$$

we abuse notation and write $\|f\|$ for both the norm on $H^2(\mathbb{D})$ and $L^2(\mathbb{T})$.

Non-tangential limits

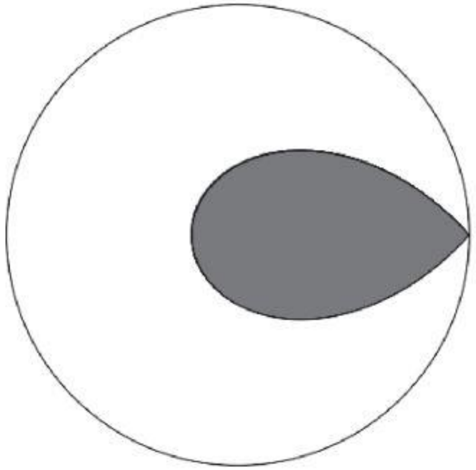
- Fatou's theorem can be extended from radial limits to *non-tangential limits*

Definition For a function $f: \mathbb{D} \to \mathbb{C}$ and $\zeta \in \mathbb{T}$, we say that $f(z)$ approaches $L \in \mathbb{C}$ **non-tangentially**, denoted

$$L = \angle \lim_{z \to \zeta} f(z),$$

if $f(z) \to L$ holds whenever $z \to \zeta$ in every fixed Stolz domain $\Gamma_\alpha(\zeta)$

$$\Gamma_\alpha(\zeta) = \{z \in \mathbb{D} : |z - \zeta| < \alpha(1 - |z|)\}, \qquad \alpha > 1$$



The Stolz domain $\Gamma_{1.5}(1)$
(Taken from Figure 1.3 of book)

Important classes of fcts in H^2

- Cauchy Kernel c_λ

$$c_\lambda(z) = \frac{1}{1 - \bar{\lambda} z}, \qquad \text{where } \lambda \in \mathbb{D}.$$

- Properties
  - $f(\lambda) = \langle f, c_\lambda \rangle$. In particular, $\|c_\lambda\|^2 = \langle c_\lambda, c_\lambda \rangle = c_\lambda(\lambda) = \frac{1}{1 - |\lambda|^2}$.
  - $\{c_\lambda : \lambda \in \mathbb{D}\}$ is linearly independent
  - $\bigvee\{c_\lambda : \lambda \in \mathbb{D}\} = H^2$
  - Let $\Lambda \subset \mathbb{D}$ have an accumulation point in $\mathbb{D}$, then $\bigvee\{c_\lambda : \lambda \in \Lambda\} = H^2$

Important classes of fcts in H^2

- Bounded analytic functions on $\mathbb{D}$, $H^\infty(\mathbb{D})$

$$H^\infty = \left\{ f\colon \mathbb{D} \to \mathbb{C} \text{ analytic} : \ \|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)| \right\}$$

We can prove $H^\infty \subset H^2$ by using the bdd integral means defn of H^2

- E.g. Möbius transformations $\tau_{\zeta,a}$ belong to H^∞

$$\tau_{\zeta,a}(z) = \zeta \frac{a - z}{1 - \bar{a}z} \qquad a \in \mathbb{D}, \zeta \in \mathbb{T}$$

$\tau_{\zeta,a}$ is a bijection from $\mathbb{D}$ to $\mathbb{D}$, and from $\mathbb{T}$ to $\mathbb{T}$.

- <u>**Definition**</u> A function $u \in H^\infty$ is an **inner function** if $|u(\zeta)| = 1$ a.e. on $\mathbb{T}$.

via radial bdry limits, which exist m-a.e. by Fatou's theorem

so $|u(z)| \le 1$ on $\mathbb{D}$ by the Max modulus theorem

Important classes of fcts in H^2

- $u \in H^\infty$ is an **inner function** if $|u(\zeta)| = 1$ a.e. on $\mathbb{T}$.

A **Blaschke product** is a function of the form

$$B(z) = \xi z^N \prod_{n=1}^{\infty} \frac{\overline{a_n}}{|a_n|} \frac{a_n - z}{1 - \overline{a_n} z}$$

where $\xi \in \mathbb{T}, N \in \mathbb{N}_0$, and $\{a_n\}_{n\geq 1} \subset \mathbb{D} \setminus \{0\}$ is s.t.

$$\sum_{n\geq 1} (1 - |a_n|) < \infty$$

- The zeros of $B(z)$ are a_n and 0 (if there is a z^N term)
- The Blaschke condition ensures that the infinite product converges nicely, and gives us $|B(\zeta)| = 1$ on $\mathbb{T}$

A **singular inner fct** is a function of the form

$$s_\mu(z) = \xi \exp\left(-\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta)\right)$$

where $\xi \in \mathbb{T}$, and $\mu \in M_+(\mathbb{T})$ with $\mu \perp m$

- $\mu \perp m$ is used to get $|s_\mu(\zeta)| = 1$ on $\mathbb{T}$
- s_μ does not vanish anywhere on $\mathbb{D}$
- $-\log|s_\mu|$ is a positive Harmonic fct …

  Herglotz representation applies … $\mathcal{P}(\mu)$ …

Important classes of fcts in H^2

- $u \in H^\infty$ is an **inner function** if $|u(\zeta)| = 1$ a.e. on $\mathbb{T}$.

Blaschke product

$$B(z) = \xi z^N \prod_{n=1}^{\infty} \frac{\overline{a_n}}{|a_n|} \frac{a_n - z}{1 - \overline{a_n} z}$$

Singular inner function

$$s_\mu(z) = \xi \exp\left(-\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta)\right)$$

__Theorem (Nevanlinna-Riesz)__ If $u \in H^\infty$ is inner, then it can be written as

$$u(z) = B(z) s_\mu(z)$$

For some Blaschke pdt B, and singular inner fct s_μ.

This factorization is unique up to a unimodular const.

Important classes of fcts in H^2

- What about factorization for $f \in H^2$?

Blaschke product

$$B(z) = \xi z^N \prod_{n=1}^{\infty} \frac{\overline{a_n}}{|a_n|} \frac{a_n - z}{1 - \overline{a_n} z}$$

Singular inner function

$$s_\mu(z) = \xi \exp\left(- \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta) \right)$$

An **outer function** is an analytic fct $F: \mathbb{D} \to \mathbb{C}$ of the form

$$F(z) = \xi \exp\left(\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \varphi(\zeta) dm(\zeta) \right)$$

where $\xi \in \mathbb{T}$, and $\varphi \in L^1$ is real-valued

Theorem (inner-outer factorization) If $f \in H^2 \setminus \{0\}$, then it can be written as

$$f = B s_\mu F$$

For some Blaschke pdt B, singular inner fct s_μ, and outer fct $F \in H^2$.

Conversely, a product of this form belongs to H^2.

This factorization is unique up to a unimodular const.

Two key operators

Unilateral shift $S: H^2 \to H^2$

Toeplitz operator $T_\varphi: H^2 \to H^2$

Key operators

- Unilateral shift operator (right shift)

$$S: H^2 \to H^2$$

$$(Sf)(z) = zf(z)$$

or

If $f = \sum_{n\geq 0} a_n z^n \in H^2$, then

$$S(a_0, a_1, a_2, \cdots) = (0, a_0, a_1, \cdots)$$

- Toeplitz operator with symbol $\varphi \in L^\infty$

$$T_\varphi: H^2 \to H^2$$

$$T_\varphi(f) = P(\varphi f)$$

Note that $T_z = S$, and $T_{\bar{z}} = S^*$

P = projection of L^2 onto H^2 (via bdry fcts)

"Riesz projection"

Key operators

$T_\varphi: H^2 \to H^2$ $\quad$ $S: H^2 \to H^2$

$T_\varphi(f) = P(\varphi f)$ $\quad$ $(Sf)(z) = zf(z)$

- Properties of Toeplitz operators
  - $aT_\varphi + bT_\psi = T_{a\varphi + b\psi}$
  - $\|T_\varphi\| = \|\varphi\|_\infty$
  - $T_\varphi = T_\psi \iff \varphi = \psi$
  - $(T_\varphi)^* = T_{\overline{\varphi}}$
  - T_φ compact $\iff \varphi = 0$

- For $\varphi, \psi \in L^\infty$, $T_\psi T_\varphi$ is a Toeplitz op $\iff T_\psi$ is conjugate analytic or T_φ is analytic.

  (i.e. if $T_\psi = T_{\overline{\sigma}}$ for some $\sigma \in H^\infty$, or if $\varphi \in H^\infty$)

  In this case, $T_\psi T_\varphi = T_{\psi\varphi}$.

- Properties of the Shift operator
  - S is an isometry
  - $\sigma(S) = \overline{\mathbb{D}}$
  - $\sigma_p(S) = \emptyset, \sigma_c(S) = \mathbb{T}, \sigma_r(S) = \mathbb{D}, \sigma_e(S) = \mathbb{T}$
  - $\sigma_p(S^*) = \mathbb{D}, \sigma_c(S^*) = \mathbb{T}, \sigma_r(S^*) = \emptyset, \sigma_e(S^*) = \mathbb{T}$

Simple eigenvalues

- **<u>Theorem (Beurling)</u>** Every non-zero subspace $\mathcal{M}$ of H^2 that is invariant under S is of the form

$$\mathcal{M} = uH^2$$

  for some inner function u. (The choice u is unique up to a unimodular constant.)

Uses inner-outer factorization, etc…

Proof outline (of Beurling's Thm)

Thm Let $0 \neq \mathcal{M} \subset H^2$, $S\mathcal{M} \subset \mathcal{M}$. Then $\mathcal{M} = uH^2$ for some inner u.

- $\mathcal{M}$ non-zero $\Rightarrow S\mathcal{M} \subsetneq \mathcal{M}$. So let us look at $\mathcal{M} \ominus S\mathcal{M} = \mathcal{M} \cap (S\mathcal{M})^\perp \neq \{0\}$
- In $\mathcal{M} \ominus S\mathcal{M}$, extract a non-zero, inner u in the following steps
  - $|u(\zeta)| \equiv$ constant m-a.e. on $\mathbb{T}$ (via a Fourier S. argument)
  - Combine ↑ with Smirnov's theorem and $u \in H^2$, to conclude $u \in H^\infty$
- Show that the S-invariant subspace generated by u, equals uH^2, i.e.

$$[u] = uH^2$$

  - ($\subseteq$) automatic, since u inner, hence uH^2 closed.
  - ($\supseteq$) approx $uG \in uH^2$ by $u\boldsymbol{G_N} \in [u]$. Uses: $G_N \to G$ in H^2 and $|u| \equiv 1$ on $\mathbb{T}$.

    Truncated Taylor series of G
- Show $[u] = \mathcal{M}$
  - ($\subseteq$) automatic, since $\mathcal{M}$ is S-invariant
  - ($\supseteq$) Use a Fourier S. argument.

$f \perp [u]$
$\Rightarrow \langle f, S^n u\rangle = 0$ and $\langle S^n f, u\rangle = 0$
$\Rightarrow f\bar{u} = 0$ a.e. on $\mathbb{T}$
$\Rightarrow f = 0$ a.e. on $\mathbb{T}$ (because u inner)

Notes
- Inner-outer factorization thm is a key ingredient for Smirnov's thm.
- In fact, the argument to extract u shows that $\mathcal{M} \ominus S\mathcal{M}$ is 1D!
- We are showing $\mathcal{M} = [\mathcal{M} \ominus S\mathcal{M}] = uH^2$

(compare this with similar results in Bergman spaces L^2_a and Dirichlet spaces $\mathcal{D}$)

Model spaces

Definition of Model Space $\mathcal{K}_u$

- **Definition** If u is an inner function, the model space $\mathcal{K}_u$ is

$$\mathcal{K}_u := (uH^2)^{\perp} = \{f \in H^2 : \langle f, uh \rangle = 0 \text{ for all } h \in H^2\}$$

- **Proposition** The model spaces $\mathcal{K}_u$ are precisely the proper S^*-invariant subspaces of H^2.
- **Proposition** via the identification with non-tangential bdry values, we have

$$\begin{aligned} \mathcal{K}_u &= H^2 \cap u\overline{zH^2} \\ &= \{f \in H^2 : f = \overline{gz}u \quad \text{a.e. on } \mathbb{T}, \text{for some } g \in H^2\} \end{aligned}$$

Reproducing kernels

- Reproducing kernel for $\mathcal{K}_u$ (depends on u and $\lambda \in \mathbb{D}$)

$$k_\lambda(z) = \left(1 - \overline{u(\lambda)}u(z)\right) c_\lambda(z) = \frac{1 - \overline{u(\lambda)}u(z)}{1 - \bar{\lambda}z}, \qquad \text{where } \lambda \in \mathbb{D}.$$

- This gives us

$$f(\lambda) = \langle f, k_\lambda \rangle, \qquad \text{where } \boldsymbol{f \in \mathcal{K}_u}.$$

- **Definition** Let P_u be the orthogonal projection of L^2 onto $\mathcal{K}_u$. (via non-tang. bdry values)

- Then, the kernels k_λ and c_λ are related by

$$k_\lambda = P_u c_\lambda$$

Special cases of $\mathcal{K}_u$

$$k_\lambda(z) = \frac{1 - \overline{u(\lambda)}u(z)}{1 - \bar{\lambda}z}$$

- If $u = z^n$, then

$$\mathcal{K}_u = (z^n H^2)^\perp = \bigvee\{1, z, \cdots, z^{n-1}\}$$

- If $u =$ finite Blaschke product with distinct zeros $\lambda_1, \cdots, \lambda_n$ with corresponding multiplicities $m_1, \cdots, m_n$, then

$$\mathcal{K}_u = \bigvee\left\{c_{\lambda_i}^{(l_i-1)} : 1 \le i \le n, 1 \le l_i \le m_i\right\}$$

(and so $\dim \mathcal{K}_u = m_1 + \cdots + m_n$).

Notes:

- $u \notin \mathcal{K}_u$!!!
- For this special case, $k_{\lambda_i} = c_{\lambda_i}$
- This formula is can be extended to infinite Blaschke pdts.
- In fact, $\dim \mathcal{K}_u < \infty \iff u =$ finite Blaschke product!

Density

- **Proposition** Let $\Lambda \subset \mathbb{D}$, and u be inner.
  - If Λ has an accumulation point in $\mathbb{D}$, then $\bigvee\{k_\lambda : \lambda \in \Lambda\} = \mathcal{K}_u$.
  - If $\sum_{\lambda\in\Lambda}(1 - |\lambda|) = \infty$, then $\bigvee\{k_\lambda : \lambda \in \Lambda\} = \mathcal{K}_u$.

Same as for c_λ in H^2

- **Proposition** $\mathcal{K}_u \cap H^\infty$ is dense in $\mathcal{K}_u$.
- **Proposition** The function S^*u generates $\mathcal{K}_u$, i.e.

$$\mathcal{K}_u = \bigvee\{S^{*n}u : n \geq 1\}$$

See Prop 8.22 of book for an improved result

Going between two model spaces

- We need to know how two inner fcts u and v are related, via the inner-outer factorization.
- **Definition** Let u and v be inner fcts.
  - $u \mid v$ means $\frac{v}{u}$ is inner
  - u and v are relatively prime, means the only common inner divisors of u and v are const fcts with unit modulus
- **Lemma** $\mathcal{K}_{u_1 u_2} = \mathcal{K}_{u_1} \oplus u_1 \mathcal{K}_{u_2}$ ... (extends to infinite products)
- **Proposition** For u, v inner functions,
  - $\mathcal{K}_u \subset \mathcal{K}_v \Leftrightarrow u|v$
  - $\mathcal{K}_u \cap \mathcal{K}_v \Leftrightarrow \mathcal{K}_{\mathrm{gcd}(u,v)}$
  - $\mathcal{K}_u \vee \mathcal{K}_v \Leftrightarrow \mathcal{K}_{\mathrm{lcm}(u,v)}$

The inner fcts $\mathrm{gcd}(u, v)$ and $\mathrm{lcm}(u, v)$ are defined in Corollary 4.8 and 4.9 of book

Boundary kernels

- We know that $f \in \mathcal{K}_u \subset H^2$ has a non-tangential limit m-a.e. on $\mathbb{T}$.
- But **for a fixed $\zeta \in \mathbb{T}$**, does $f \in \mathcal{K}_u$ have a non-tangential limit?
- This is related to the existence of the limit of

$$k_\lambda(z) = \frac{1 - \overline{u(\lambda)}u(z)}{1 - \bar{\lambda}z} \in \mathcal{K}_u \cap H^\infty \subset H^2$$

See Thm 7.24 of book

as we take $\lambda \to \zeta \in \mathbb{T}$. The limit (if it exists) is called the **boundary kernel** $k_\zeta \in \mathcal{K}_u$.

- Does the limit k_ζ satisfy the reproducing kernel property at $\mathbb{T}$

$$f(\zeta) = \langle f, k_\zeta \rangle \, ?$$

- It turns out that boundary kernels are the eigenvectors of an important class of ops on $\mathcal{K}_u$, called Clark unitary operators. See Thm 11.4 of book.

See Thm 11.4 of book

The compressed shift

$P_u: L^2 \to \mathcal{K}_u$

The compressed shift (definition)

- **Definition** The compressed shift is the operator $S_u: \mathcal{K}_u \to \mathcal{K}_u$ defined by

$$S_u f = P_u S f$$

- $S_u: \mathcal{K}_u \to \mathcal{K}_u$ is a compression of $S: H^2 \to H^2$ to the subspace $\mathcal{K}_u$, that is,

$$p(S_u) = P_u p(S)\Big|_{\mathcal{K}_u}$$

for all analytic polys p (i.e. no $\bar{z}^n$)

The compressed shift (properties)

- **Proposition** The following identity holds
  - $S_u = CS_u^*C$. ("the compressed shift is a complex symmetric op")
  - $I - S_uS_u^* = k_0 \otimes k_0$
  - $I - S_u^*S_u = Ck_0 \otimes Ck_0 = S^*u \otimes S^*u$
- **Proposition** S_u^n and S_u^{*n} converges to the zero op. in the SOT. That is,

$$\|S_u^n f\| \to 0 \text{ and } \|S_u^{*n} f\| \to 0, \qquad as\ n \to \infty, \text{for all } f \in \mathcal{K}_u .$$

- **Proposition** $\mathcal{K}_u = \bigvee\{S_u^{*n}k_0 : n \geq 0\}$
- **Theorem** The compressed shift S_u is irreducible. That is, there are no proper non-trivial reducing subspace for $\mathcal{K}_u$ (invariant for both S_u and S_u^*).

Notes:

- **Defn** (Conjugation on $\mathcal{K}_u$)

$C: \mathcal{K}_u \to \mathcal{K}_u$ with $Cf = \overline{fz}u$

(via bdry fcts)

- $I - SS^* = c_0 \otimes c_0$
- $I - S^*S = 0$
- $\mathcal{K}_u = \bigvee \{S^{*n}u : n \geq 1\}$
- Think CNU contractions.

Key theorems

- **Theorem** (Wold Decomposition) Let $T \in \mathcal{B}(\mathcal{H})$ be a contraction on a Hilbert space $\mathcal{H}$.

  Then we may write (uniquely!)

$$T = U \bigoplus K$$

U: a unitary op

K: a completely non-unitary contraction (i.e. no non-trivial reducing subspace for K)

- **Theorem** (Sz.-Nagy-Foiaș) Let T be a contraction on a Hilbert space $\mathcal{H}$ such that
  - $\|T^{*n}x\| \to 0$ as $n \to \infty$, for all $x \in \mathcal{H}$,
  - $\operatorname{rank}(I - T^*T) = \operatorname{rank}(I - TT^*) = 1$,

  Then, there exists an inner fct u such that $T: \mathcal{H} \to \mathcal{H}$ is unitarily equiv to $S_u: \mathcal{K}_u \to \mathcal{K}_u$.

Proof outline (of Sz.-Nagy-Foiaș)

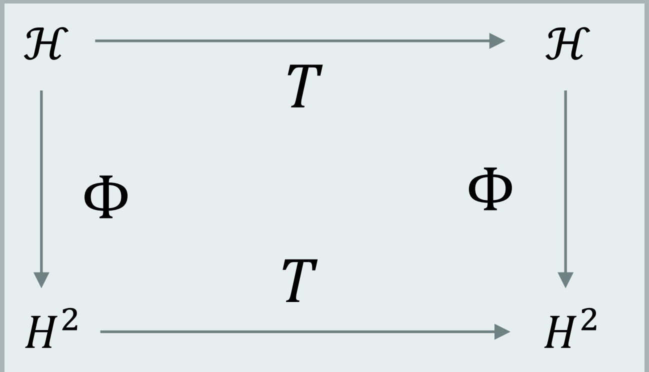
Thm Let $T \in \mathcal{B}(\mathcal{H})$ with $\|T\|_{op} \leq 1, T^{*n} \to 0$ in the SOT, and $\text{rank}(I - T^*T) = \text{rank}(I - TT^*) = 1$, then $\exists$ inner u such that $T \cong S_u$.

- Construct the defect operator $D = \sqrt{I - T^*T}$, via the spectral thm of the (positive) SA operator $I - T^*T$.
- Embed $\mathcal{H}$ into H^2 using the isometry (hypothesis $T^{*n} \to 0$ in SOT used here)

$$\Phi: \mathcal{H} \to H^2 \cong l^2(\mathbb{N}_0)$$

$$\Phi \text{x} = (Dx, DTx, DT^2x, \cdots)$$

- range(Φ) is S^*-invariant. So by Beurling's theorem, range(Φ) $= \mathcal{K}_u$ for some inner u.
- Check



, then restrict Φ's codomain from H^2 to $\mathcal{K}_u$ to get a unitary operator

- Convert from S_u^* back to S_u using $US_uU^* = S_{u^\#}^*$

H^∞-functional calculus

H^∞ - functional calculus

- Fix an inner function u.

- **Definition** The H^∞-functional calculus for S_u is the mapping

$$\Lambda: H^\infty \to \mathcal{B}(\mathcal{K}_u)$$
$$\varphi \mapsto \varphi(S_u) := P_u T_\varphi \Big|_{\mathcal{K}_u}$$

i.e. $\varphi(S_u)f = P_u(\varphi f)$, for $f \in \mathcal{K}_u$.

Basic properties

$\Lambda: H^\infty \to \mathcal{B}(\mathcal{K}_u)$

$\varphi \mapsto \varphi(S_u) := P_u T_\varphi \big|_{\mathcal{K}_u}$

- **Theorem** Fix an inner fct u. Then the mapping $\Lambda: H^\infty \to \mathcal{B}(\mathcal{K}_u)$ is
  - linear, multiplicative, a contraction (i.e. $\|\varphi(S_u)\| \leq \|\varphi\|_\infty$), and $\Lambda z = S_u$.

an algebra homomorphism (but not a *-alg homo)

- **Theorem** Fix an inner fct u and $\varphi \in H^\infty$. Then,
  - $\varphi(S_u)^* = T_{\overline{\varphi}}|_{\mathcal{K}_u}$
  - If $\sum_{n\geq 0}|\hat{\varphi}(n)| < \infty$, then $\varphi(S_u) = \sum_{n\geq 0} \hat{\varphi}(n) S_u^n$ (conv in op norm)
  - $\varphi(S_u) = 0$ iff $\varphi \in uH^\infty$ (the symbol is not unique! Unlike Toeplitz ops.)

be careful with adjoints! $\overline{\varphi} \notin H^\infty$

- **Theorem** Fix an inner fct u and $\varphi \in H^\infty$. Let $\{\varphi_n\}_{n\geq 1} \subset H^\infty$ be s.t. $\sup_{n\geq 1}\|\varphi_u\|_\infty < \infty$.
  - If $\lim_{n\to\infty} \varphi(\zeta) = \varphi(\zeta)$ a.e. on $\mathbb{T}$, then $\varphi_n(S_u) \to \varphi(S_u)$ in the SOT.
  - If $\lim_{n\to\infty} \varphi(z) = \varphi(z)$ for all $z \in \mathbb{D}$, then $\varphi_n(S_u) \to \varphi(S_u)$ in the WOT.

The spectrum of S_u

- <u>Definition</u> Let $u = Bs_\mu$ be a non-constant inner fct. The **spectrum of $\boldsymbol{u}$, $\boldsymbol{\sigma}(\boldsymbol{u})$** is the set

$$\sigma(u) = \overline{\{a_n\}}_{n\geq 1} \cup \operatorname{supp} \mu$$

The zeros of B, a_n, lie in $\mathbb{D}$ and may accumulate on $\mathbb{T}$

subset of $\mathbb{T}$

- <u>Theorem</u> **(Livšic-Möller)** $\sigma(S_u) = \sigma(u)$

- <u>Corollary</u> $\sigma_p(S_u) = \sigma(u) \cap \mathbb{D} = \{\lambda \in \mathbb{D} : u(\lambda) = 0\}$. The eigenvalues are simple.

- <u>Proposition</u> $\sigma_e(S_u) = \sigma(u) \cap \mathbb{T}$

Some operator algebraic properties

- We need some vocabulary from operator algebras.
- **Definition** Let $C^*(S_u)$ be the unital C^*-algebra generated by S_u
- **Definition** $\mathcal{C}\big(C^*(S_u)\big)$ = smallest norm closed two-sided ideal of $\mathcal{B}(\mathcal{H})$ containing all commutators

$$AB - BA, \qquad \text{where } A, B \in C^*(S_u).$$

- **Theorem** For u inner, we have
  - $C^*(S_u) = \{\varphi(S_u) + K : \varphi \in C(\mathbb{T}) \text{ and } K\colon \mathcal{K}_u \to \mathcal{K}_u \text{ compact}\}$
  - $\mathcal{C}\big(C^*(S_u)\big) = \{\text{compact ops in } \mathcal{K}_u\}$
  - $\dfrac{C^*(S_u)}{\{\text{compact ops in } \mathcal{K}_u\}} \cong C(\sigma(u) \cap \mathbb{T})$ as C^*-algebras

Need to first make sense of $\varphi(S_u)$ for a symbol $\varphi \in C(\mathbb{T})$ as opposed to $\varphi \in H^\infty(\mathbb{D})$

c.f. Gelfand-Naimark thm

The spectrum of $\varphi(S_u)$

- **Theorem** (spectral mapping) Let u be inner and $\varphi: \mathbb{D} \to \mathbb{C}$ be analytic with a continuous extension to $\overline{\mathbb{D}}$, then
  - $\sigma(\varphi(S_u)) = \varphi(\sigma(S_u)) = \varphi(\sigma(u))$
  - $\sigma_e(\varphi(S_u)) = \varphi(\sigma_e(S_u)) = \varphi(\sigma(u) \cap \mathbb{T})$
- **Theorem** Let u be inner and $\varphi \in H^\infty$. Then

$$\sigma(\varphi(S_u)) = \left\{\lambda \in \mathbb{C} : \inf_{z \in \mathbb{D}} (|u(z)| + |\varphi(z) - \lambda|) = 0\right\}$$

- **Theorem** (point spectrum) Let u be inner and $\varphi \in H^\infty$. Fix $\lambda \in \mathbb{C}$. Set

$$v = \gcd((\varphi - \lambda)_{\text{inner}}, u)$$

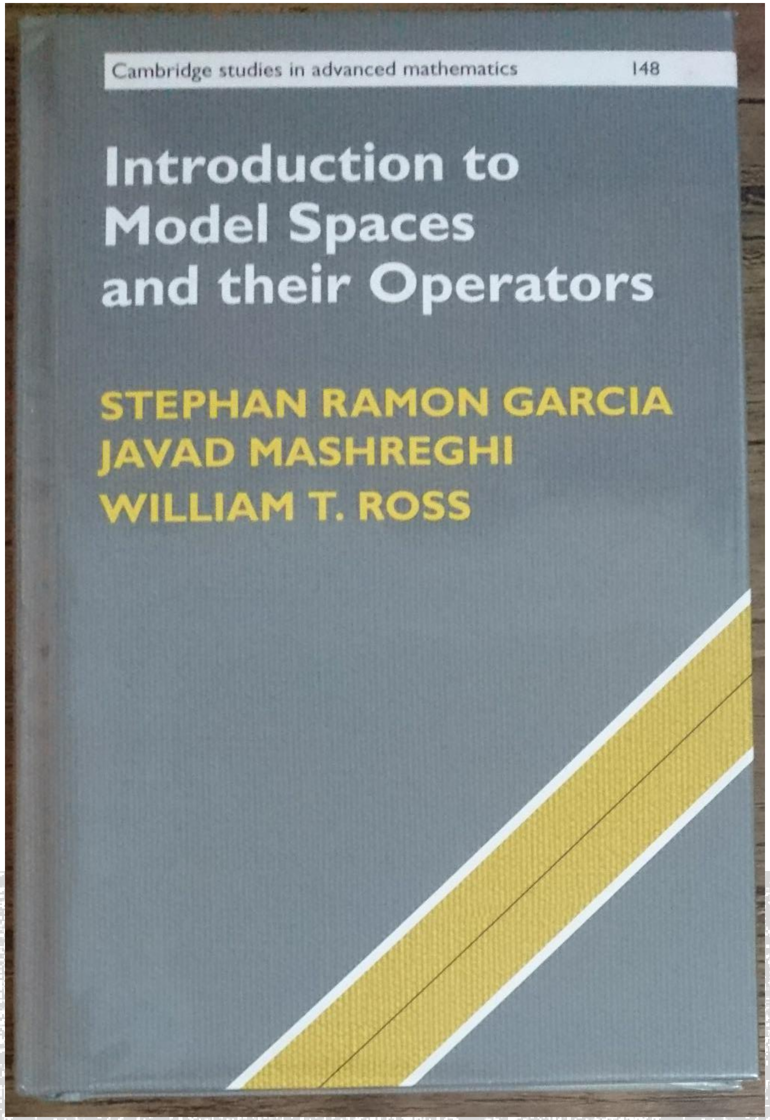
Then

$$\ker(\varphi(S_u) - \lambda) = \frac{u}{v}\mathcal{K}_v \quad \text{and} \quad \ker(\bar{\varphi}(S_u) - \bar{\lambda}) = \mathcal{K}_v$$

So, $\lambda \in \sigma_p(\varphi(S_u)) \iff \bar{\lambda} \in \sigma_p(\bar{\varphi}(S_u)) \iff v = \gcd((\varphi - \lambda)_{\text{inner}}, u)$ is not constant

Notes:
- Livšic-Möller gives $\sigma(S_u) = \sigma(u)$
- The statement on $\sigma_e(\varphi(S_u))$ needs some operator algebra machinery
- Compare with: $\sigma(S_u) = \sigma(u) = \left\{\lambda \in \overline{\mathbb{D}} : \liminf_{z \to \lambda} |u(z)| = 0\right\}$
- Need to define $\bar{\varphi}(S_u)$ first (c.f. "Truncated Toeplitz ops")

Thank you!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Introduction to Model Spaces and their Operators
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